A GENERAL GEOMETRIC MODEL FOR DETERMINING OPTIMAL SHAPES OF PLANAR ELASTIC PIVOTS

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INTRODUCTION

Elastic (flexural) pivots are commonly used in precision machines and instruments as bearings for guiding small displacements and rotations [1]. They provide accurate motion without friction and with very little hysteresis. Elastic pivots, such as those illustrated in Fig. 1, are often designed using straight beams or conic sections with circular, elliptical, parabolic, or hyperbolic shapes. Previously, each of these shapes required separate analyses using different analytical equations or finite element analysis (FEA). In this paper, we describe a general geometric model capable of representing all these shapes using two-dimensional, quadratic, rational Bézier curves. This mathematical representation of the geometry is well suited for computer aided design, analysis (either analytical or FEA), and optimization studies.

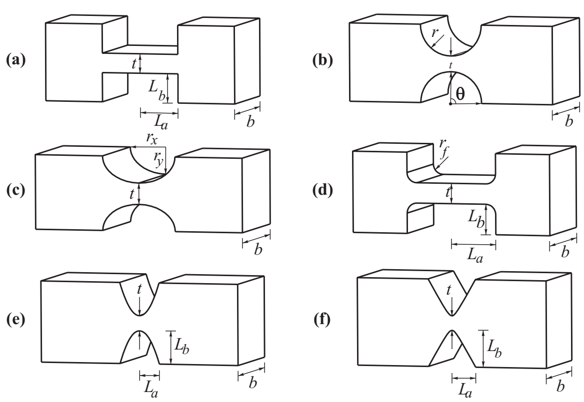

FIGURE 1. Elastic pivot shapes and their commonly used dimensions, (a) leaf spring, (b) circular, (c) elliptical, (d) corner-filleted, (e) parabolic, and (f) hyperbolic

The performance of elastic pivots depends on three crucial aspects: compliance (or its inverse stiffness), accuracy of motion (e.g. rotation without translation), and the amount of stress resulting from elastic deformation. An ideal elastic pivot would have high compliance in the intended direction of rotation and infinite stiffness in the remaining directions. Since this is not possible, designers must compromise on attributes like static stiffness, resonant frequencies, and lost motion.

Analytical estimates of compliance for several pivot shapes are available [2, 3, 4]. However, fewer analytical approaches are available for predicting the stress within the pivot, which is necessary for determining the pivots range of rotation. Therefore, it is common for designers to use analytical models to estimate compliance and then use FEA to validate the compliance, determine the stress, and determine motion range.

This design method is not ideal. Since analytical models are unique to each pivot shape, designers must consider alternate shapes independently and consecutively. Many designers therefore only consider a single shape, despite having analytical models of compliance for a variety of shapes. Those that consider alternate shapes often find that a single best choice is not obvious, especially when the designer considers several design parameters and performance factors (e.g. high resonant frequencies in multiple directions, static stiffness, manufacturing constraints, etc.). It would therefore be beneficial to have a unified geometric model applicable for most pivot shapes that is compatible with analysis techniques and optimization. This paper presents such a model, and it was recently applied and demonstrated by Haghighian [5].

QUADRATIC RATIONAL BÉZIER CURVES

Circles, ellipses, parabolas, and hyperbolas are all conic sections generated by the intersection of a plane with a cone. When these curves lie in a two-dimensional Cartesian plane, the coordinates of points along the curve satisfy an implicit equation. The explicit form of the equation gives the y-coordinate of points along the curve as a function of the independent x-coordinate. Explicit equations are typically used by others when deriving analytical estimates of compliance for elastic pivots. Alternatively, conic sections can be specified with parametric equations, such as Eq. 1 for a circle. This is more convenient for curve evaluation since the independent variable is the parametric variable $0 \leqq s \leqq 1$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_o \\ y_o \end{bmatrix} + r \begin{bmatrix} cos(s\theta) \\ sin(s\theta) \end{bmatrix} \tag{1}$$

A single parametric equation can represent all of the conic sections. This is achieved with quadratic rational Bézier curves [6], which exactly represent conic sections and other shapes via the ratio of polynomial functions. The coordinates $[x, y]$ of points along a two dimensional rational Bézier curve of order n are computed with Eq. 2 in terms of the parametric variable s, the coordinates of the $n+1$ control points $\mathbf{P}_i$, and the $n+1$ scalar weights w_i that are applied to each control point. The parametric variable s varies between zero and one, and the Bernstein polynomials $B_{i,n}$, given by Eq. 3.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\sum_{i=0}^{n} B_{i,n}(s)\, w_i \mathbf{P}_i}{\sum_{i=0}^{n} B_{i,n}(s)\, w_i} \tag{2}$$

$$B_{i,n}(s) = \frac{n!}{i!\,(n-i)!} s^i (1-s)^{n-i} \tag{3}$$

When the curve is quadratic (three control points and $n = 2$), Eq. 2 is expanded using Eq. 3 to obtain Eq. 4. It is sometimes convenient to divide the coefficients in the numerator by the denominator to form $n+1$ rational blending functions $R_{i,n}(s)$ since these directly blend the coordinates of the control points, as shown in Eq. 5.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{w_0 (1-s)^2 \mathbf{P}_0 + 2w_1 s (1-s) \mathbf{P}_1 + w_2 s^2 \mathbf{P}_2}{w_0 (1-s)^2 + 2w_1 s (1-s) + w_2 s^2} \tag{4}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = R_{0,2}(s)\,\mathbf{P}_0 + R_{1,2}(s)\,\mathbf{P}_1 + R_{2,2}(s)\,\mathbf{P}_2 \tag{5}$$

Quadratic rational Bézier curves therefore use three control points, $\mathbf{P}_i = [x_i, y_i, z_i]$ where $i = 0, 1$, or 2, to regulate their shape. The curve starts at control point $\mathbf{P}_0$ when $s = 0$, and it ends at control point $\mathbf{P}_2$ when $s = 1$. The tangency of the curve at $\mathbf{P}_0$ is parallel to $\mathbf{P}_1 - \mathbf{P}_0$, and the tangency of the curve at $\mathbf{P}_2$ is parallel to $\mathbf{P}_2 - \mathbf{P}_1$. The control polygon (triangle for second order with $n = 2$) for the curve is formed by line segments L_i that connect the control points.

Figure 2(a) shows an example that is applicable to elliptical pivots [3]. The elliptical pivot has dimensions $t = 0.002$ m, $r_x = 0.005$ m, and $r_y = 0.003$ m. It is specified with three control points $\mathbf{P}_1 = [0, t/2] = [0, 0.001]$, $\mathbf{P}_2 = [r_x, t/2] = [0.005, 0.001]$, and $\mathbf{P}_3 = [r_x, t/2 + r_y] = [0.005, 0.004]$ and three weights, where $w_0 = w_2 = 1$ and $w_1 = \sqrt{2}/2$. Figure 2(b) also shows plots of the curve's rational functions $R_{i,n}(s)$ as used in Eq. 5. The specification of one quarter of a circle is a special case of an ellipse that is obtained in a similar fashion, with the weight $w_1 = \sqrt{2}/2$.

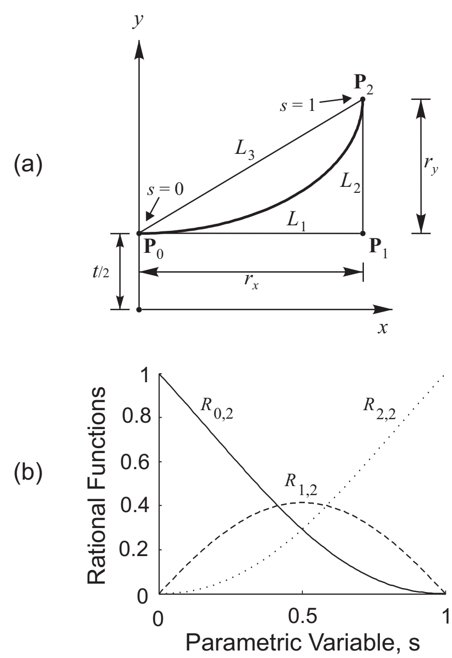

FIGURE 2. (a) Elliptical curve for elastic pivot, specified as a quadratic rational Bézier curve with three control points and its (b) rational blending functions

GENERAL MODEL

A general geometric model for half of an elastic pivot is constructed using the parametric equations for quadratic rational Bézier curves.

Figure 3 shows the model, and it consists of twelve curve segments c_i. Curve segments c_3 and c_{10} are quadratic rational Bézier curves, and all others are simple line segments with Bézier representations. The origin of a Cartesian coordinate system is located at the center of the pivot so that the two planes of symmetry coincide with the x-z and y-z planes. The shape of the pivot is governed by fifteen control points $\mathbf{P}_1$ through $\mathbf{P}_{15}$ and their corresponding weights w_1 through w_{15}. The coordinates of these control points and their weights are collected within a control points matrix cp as shown in Eq. 6.

$$cp = \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 & \dots & \mathbf{P}_{15} \\ w_1 & w_2 & \dots & w_{15} \end{bmatrix} \quad (6)$$

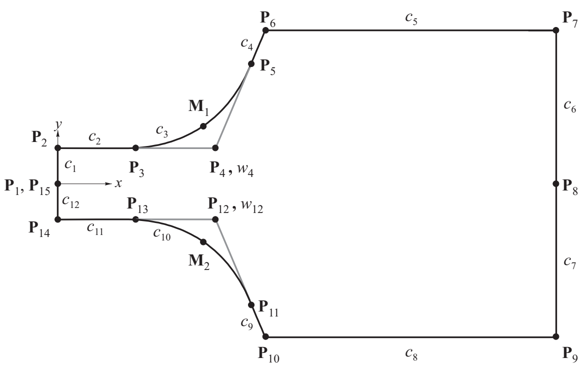

FIGURE 3. A general geometric model for the perimeter of half of an elastic pivot, based on a set of line and curve segments specified by fifteen control points and weights.

All of the pivots illustrated in Fig. 1 can be constructed using this general model, but some of these pivots do not require all of the line or curve segments. The leaf spring does not require c_3 and c_{10}, and the circular, elliptical, parabolic, and hyperbolic pivots do not require c_2, c_4, c_9 ,and c_{11}. The corner-filleted pivot requires all of the line and curve segments. Therefore, a shape vector $\mathbf{S}$, given in Eq. 7, accounts for whether each curve c_i is included in the pivot's shape. An element s_i corresponds to segment c_i, and its value equals zero if the segment is not present within the shape, one if the segment is a line, and two if the segment is a quadratic rational Bézier curve. Although quadratic rational Bézier curves can represent lines, it is sometimes useful to distinguish between the two. As an example, the shape vector $\mathbf{S}$ for an elliptical pivot in which segments c_2, c_4, c_9, c_{11} are not present and c_3 and c_{12} are curves is given by $\mathbf{S} = [102011110201]^T$.

$$\mathbf{S} = [s_0, s_1, s_2, ..., s_{15}]^T \quad (7)$$

The coordinates of the 15 control points shown in Figure 3 and collected in cp are determined from a set of design parameters that include the following:

- t, the thickness of the pivot in the y-z plane,
- L_1, the length of line segment c_2,
- $\mathbf{T}_3 = [T_3^x, T_3^y]$, the tangency vector at control point $\mathbf{P}_3$,
- $\mathbf{M} = [M^x, M^y]$, some point along the middle of curve segment c_3,
- $\mathbf{P}_5 = [P_5^x, P_5^y]$, a position vector locating control point $\mathbf{P}_5$,
- $\mathbf{T}_5 = [T_5^x, T_5^y]$, the tangency vector at control point $\mathbf{P}_5$,
- L_2, the length of line segment c_4, and
- L_3, the length of line segment c_5.

These design parameters are collected within a design vector $\mathbf{X}$ as shown in Eq. 8, which consists of twelve scalar dimensions, coordinates, or components of the tangent vectors.

$$\mathbf{X} = [t, L_1, T_3^x, T_3^y, M^x, M^y, P_5^x, P_5^y, T_5^x, T_5^y, L_2, L_3] \quad (8)$$

The common shapes shown in Fig. 1 require that some of the design parameters be constrained to particular values. Table 1 lists the form of the shape vector $\mathbf{S}$ and gives equations for determining the constrained values of the design parameters for the common shapes. The tangency vector $\mathbf{T}_3$ equals $[1, 0]$ so that continuity in the first derivative is satisfied at control points $\mathbf{P}_2$ or $\mathbf{P}_3$. The components of $\mathbf{M}$, $\mathbf{P}_5$ and $\mathbf{T}_5$ are found by rearranging the implicit equations of the conic sections.

IMPLEMENTATION FOR COMPUTER AIDED DESIGN AND ANALYSIS

The geometric model was implemented by Haghighian [5] within a toolbox of scripts that run within Matlab™ and enable the computer aided design, analysis, and optimization of elastic pivots. It is possible to model the shape of elastic

TABLE 1. Values of design parameters for common pivot shapes

DP	Circle	Ellipse	Corner Filleted	Parabolic
$\mathbf{S}$	$[102011110201]^T$	$[102011110201]^T$	$[112111111211]^T$	$[102011110201]^T$
L_1	–	–	$L_a - r_f$	–
T_1^x	1	1	1	1
T_1^y	0	0	0	0
M^x	$r\sin\left(\frac{\theta}{2}\right)$	$r_x\cos\left(\frac{\pi}{4}\right)$	$L_1 + r_f\cos\left(\frac{\pi}{4}\right)$	$\frac{L_a}{2}$
M^y	$\frac{t}{2} + r\left(1-\cos\left(\frac{\theta}{2}\right)\right)$	$\frac{t}{2} + r_y\left(1-\sin\left(\frac{\pi}{4}\right)\right)$	$\frac{t}{2} + r_f\left(1-\sin\left(\frac{\pi}{4}\right)\right)$	$\frac{3t}{8} + \frac{L_b}{4}$
P_5^x	$r\sin(\theta)$	r_x	$L_1 + r_f$	L_a
P_5^y	$\frac{t}{2} + r(1-\cos(\theta))$	$\frac{t}{2} + r_y$	$\frac{t}{2} + r_f$	L_b
T_5^x	$\sin\left(\frac{\pi}{2}-\theta\right)$	1	1	1
T_5^y	$\cos\left(\frac{\pi}{2}-\theta\right)$	0	0	$\frac{1}{L_a}(2L_b - t)$
L_2	–	–	$L_b - r_f$	–

pivots in four possible representations. The first representation uses the dimensions of the common pivot shapes as illustrated in Figure 1. A function named 'fptptodp' converts these dimensions into values of the design parameters using Table 1. It is then possible to convert from the design parameter representation to coordinates and weights of the control points using the 'fpdptocp' function or vice versa using 'fpcptodp'. Finally, the coordinates of points around the perimeter of the elastic pivot can be determined using the 'fpcptocoords' function.

Plane strain or plane stress FEA is automatically performed by the function 'fpfemlab2d', which builds the pivot shape within FEMLAB TM(COMSOL AB), a finite element program that operates within MATLAB. The script performs three analyses in order to determine a 2D compliance matrix C. The three analyses subject the pivot to an axial load, transverse load, and pure bending, from which the first, second, and third columns of the compliance matrix are determined, respectively. An additional function provides for superposing these results to determine stress within the pivot when simultaneously subjected to combined axial, transverse, and bending loads. Haghighian [5] validated results from our FEA method against others' analytical/fea results, and he also demonstrated the optimization of a pivot's shape using non-linear, constrained optimization techniques that employ the results of finite element analyses (e.g. compliance, deflection, and stress) within the objective function and/or constraints.

CONCLUSIONS

A general geometric model, based on Bézier curves, is described for representing the shapes of most common two-dimensional elastic pivots. The model is implemented in a set of Matlab/COMSOL scripts that permit the design and analysis of pivot shapes in an automated fashion. Although not presented in this paper, Haghighian [5] demonstrated the use of this approach in optimizing pivot shapes for particular load conditions within flexural mechanisms. This work will be presented in a future article.

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